Solving some questions on Topological Games

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Abstract

We solve some questions presented in Gilton and Holshowser [GH24].

0 Introduction

This paper is written as a response to Gilton and Holshowser's paper [GH24], in which he poses 5 questions related to the preservation of topological games (more specifically, preservation of the winning strategies of topological games).

The forcing notion $\mathbb{P}$ is assumed to be strongly proper unless specified otherwise. Also, the preservation of a topological property in this paper will be done using strongly proper forcings unless said otherwise.

1 Question 1

To prove (8.1) in Gilton, we need the following facts:

1. If $\mathbb{P}$ is strongly proper for stationarily many models, then $\mathbb{P}$ preserves player II having a winning strategy on $G_\square(\mathcal{O}_X, \mathcal{O}_X)$. More precisely, If $\mathbb{P}$ is strongly proper for stationarily many models, then $\mathbb{P}$ preserves (forces) that, for a given θ (which is a large enough regular cardinal), and that $(X, \tau) \in H(\theta)$, for which countable M embedded in $H(\theta)$, is another space for which II has a winning strategy on $G_\square(\mathcal{O}_X, \mathcal{O}_X)$, then $\mathbb{P}$ forces that II has a winning strategy on $G_\square(\mathcal{O}_X, \mathcal{O}_X)$.

2. II has a winning strategy for $G_\square(\otimes_X, \otimes_X)$ iff Ω_p^2 is countable.

3. Ω-Menger properties are preserved by $\mathbb{P}$.

4. Menger-ness is preserved by a strongly proper forcing (and even a Cohen forcing!) for a topological space X.

5. All blades are in $H(\theta)$.

1.1 Remarks on the cardinal θ for Question 1

Lemma 1. θ is weakly Mahlo.

Open Question 1. Is (or can) θ be a large cardinal that is stronger than weak inaccessibility?

Open Question 2. Is the "Rothberger Axiom" (named this because Rothbergerness is independent from ZFC) consistent with inaccessibility?

These open questions result from curiosity about the nature of θ.

2 Question 2

Here are some remarks related to Question 2:

1. The winning strategy in the Menger game from II is a function $\Phi : O^{<\omega} \to \tau$, with $\Phi(\langle V_0..V_n\rangle) \in F_n$ for each $\langle V_0..V_n\rangle$ in $O^{<\omega}$, in which $O^{<\omega}$ is a set of open covers of X. (notation from [AD19].)

2. $\mathbb{P}$ preserves mengerness, given for M stationary in $[H(\theta)]^{\aleph_0}$.

Open Question 3. Does $\mathbb{P}$ preserve mengerness for M NOT stationary in $[H(\theta)]^{\aleph_0}$?

Generally, with most preservation theorems we have $\mathbb{P}$ preserve mengerness for M stationary in $[H(\theta)]^{\aleph_0}$, but there may be some exceptions.

2.1 Proof of Question 2

Proof. Let p_0 be a condition in $\mathbb{P}$. Fix it and a sequence of open covers in $\langle F_0, ..., F_n \rangle$, and take the sequence $\langle \dot{F}_n : n \in \omega$ of $\mathbb{P}$-names for $\langle F_0, ...F_n \rangle$. The rest proceeds as essentially how Gilton proceeds when proving that a strongly proper $\mathbb{P}$ preserves Mengerness for topological spaces, but the extension $q \leq p$ and sequence $\langle \dot{F}_n : n \in \omega$ of $\mathbb{P}$-names now satisfy:

1. For each $n \in \omega$, $q \Vdash \dot{F}_n$ is a non-empty finite subset of $\dot{V}_n$ of X, an open cover of X.
2. $\bigcup_{n \in \omega} F_n$ is a cover of X.

□

3 Question 3

As a remark, note that Cohen forcing (with the measure algebra) already preserves Mengerness. More precisely, Mengerness is preserved for a notion of forcing that is weakly endowed. An *endowed* notion of forcing is a notion of forcing such that if there is a decomposition of $\mathbb{P}$ into an increasing union of length ω, say $\mathbb{P} = \bigcup_{n \in \omega} P_n$ in which $P_n \subseteq P_{n+1}$ $\forall n$, and a sequence $\langle L_n : n \in \omega$ of sets satisfying the conditions in [Kad10].

The proof of Question 3 goes similar to the proof of Question 2, but with Cogen forcing instead.

4 Question 4

This is an immediate result of Question 1; since Question is true, 4 is true also.

5 Question 5

A winning strategy for II on a k-Rothberger game on a topological space X is a function like the one described in [AD19], but with open covers replaced with k-covers (Same thing also applies for k-Menger games).

Note that many true properties are preserved and also true for k-covers, making the probability of this question also being true very high, and in fact is it true for Mengerness, but k-Rothbergerness is uncertain.

Theorem 2. II's winning strategy for the k-Menger game on X is preserved.

The proof proceeds like the proof of Question 2, just with k-covers instead of open covers.

6 Additional Remarks

Open Question 4. Can Cohen Forcing preserve II's winning strategy for the k-Menger game? k-Rothberger?

Open Question 5. Do other forcings preserve II's winning strategy for topological games?

$H(\theta)$ and $R(\theta)$ are very similar. Preservation of Rothbergerness (with a strongly proper forcing) using $R(\theta)$ is possible, espescially considering that $H(\theta) \subseteq R(\theta)$ and that the two share so many properties, but what about other topological properties and games?

First, we draw off of the terminology from [Sch10]. Rothberger spaces are indestructibly Lindelof.

Open Question 6. Is it possible that if it is consistent that there is a measurable cardinal, then it is consistent that II has a winning strategy in the k-Rothberger game? This question was inspired off of [ST09].

Open Question 7. Can the process used in solving Question 2, 4, and (part) of 5 be streamlined, and therefore used in other Topological games? What about other objects in Game Theory, Set Theory or Topology?

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References

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